Math 120A
Differential Geometry

Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

(a) [4pts.] Define an orientable surface.

> **Solution:** A surface S is said to be orientable if there is an atlas $\mathcal{A}$ such that if Φ is a transition map between any two surface patches for S in $\mathcal{A}$ and $J(\Phi)$ is the Jacobian of Φ, then $\det(J(\Phi)) > 0$.

(b) [5pts.] Prove that the torus with the parametrization

$$\sigma(\theta, \phi) = ((a + b\cos\theta)\cos\phi, (a + b\cos\theta)\sin\phi, b\sin\theta)$$

applied to appropriate domains is an orientable surface. [Hint: You can argue this without doing a computation.]

> **Solution:** Using the usual atlas for the torus (the parametrization above applied to suitably restricted domains), all transition maps are the identity or a shift by $2\pi n$ in one or both coordinates, and therefore their Jacobians have determinant one everywhere.

(c) [1pts.] Comment on what this means about the shape of the torus.

> **Solution:** Anything like "the torus divides $\mathbb{R}^3$ into a distinct interior and exterior", "the torus has two sides that can be painted different colors", etc.

Problem 2.

(a) [5pts.] Define an allowable surface patch for a smooth surface S.

> **Solution:** We say $\sigma : U \to S \cap W$ is an allowable surface patch for S if U is an open set in $\mathbb{R}^2$, W is an open set in $\mathbb{R}^3$, and σ is a homeomorphism which is smooth and has linearly independent partial derivatives at all points in its domain.

(b) [5pts.] Let $f(x, y)$ be any smooth function from an open set $U \subset \mathbb{R}^2$ to $\mathbb{R}$. Prove that the graph $S = \{(x, y, f(x, y))\}$ is a smooth surface covered by a single allowable surface patch, and find an the tangent plane of f at an arbitrary point $(x_0, y_0, f(x_0, y_0))$.

> **Solution:** The map $\sigma : U \to \mathbb{R}^3$ given by $(x, y) \to (x, y, f(x, y))$ is clearly smooth and injective with continuous inverse (projection to the xy-plane), and the image of σ is all of S. The partial derivatives of σ are $\sigma_x = (1, 0, f_x)$ and $\sigma_y = (0, 1, f_y)$, and their cross product $\sigma_x \times \sigma_y = (-f_x, -f_y, 1)$ is never zero. So σ is an allowable surface patch and S is a surface. An equation for the tangent plane at $(x_0, y_0, f(x_0, y_0))$ is $f_x(x_0, y_0)x + f_y(x_0, y_0)y - z = 0$.

Problem 3.

(a) [5pts.] Define a quadric.

> **Solution:** A quadric is the solution set to an equation $\mathbf{v}^t A\mathbf{v} + \mathbf{b}^t\mathbf{v} = c$, where A is a symmetric 3×3 matrix, $\mathbf{b}$ is a constant vector, and c is a constant scalar.

(b) [5pts.] Verify that the solution set S to $16x^2 + 24xy + 9y^2 + 25x = 1$ is a smooth surface. What type of surface is it?

> **Solution:** If $f(x, y, z) = 16x^2 + 24xy + 9y^2 + 25x - 1$, the gradient of f is $\nabla f = (32x + 24y + 25, 24x + 18y, 0)$, which is never zero. So since S is the level set $f(x, y, z) = 0$, S is a smooth surface. To classify it, we diagonalize
>
> $$A = \begin{pmatrix} 16 & 12 & 0 \\ 12 & 9 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
>
> Inspection or computation shows the eigenvectors are $(0, 0, 1)$ with eigenvalue 0, $(-3, 4, 0)$ with eigenvalue 0, and $(4, 3, 0)$ with eigenvalue 25. Ergo we diagonalize by
>
> $$P = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
>
> and obtain the matrix PAP^t
>
> $$A = \begin{pmatrix} 25 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
>
> Therefore after multiplication by P our quadric is the solution set to $25x^2 + 20x - 15y = 1$. We complete the square, obtaining $(25x^2 + 20x + 4) - 15y = 5$, or $(5x + 2)^2 = 5 + 15y$. After translations this is a parabolic cylinder.

Problem 4.

(a) [5pts.] Define a vertex of a regular simple closed curve $\gamma(t)$ in $\mathbb{R}^2$.

> **Solution:** Let κ_s be the signed curvature function of γ. The curve is said to have a vertex anywhere $\dot{\kappa}_s = 0$.

(b) [5pts.] Show that there is no simple closed curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ with turning angle $\phi(t)$ equal to a polynomial of degree > 1. [Hint: What is $\ddot{\phi}(t)$?]

Solution: Recall that a simple closed curve has at least four vertices. So in particular if we have γ simple closed defined on all of $\mathbb{R}$, then $\dot{\kappa}_s(t) = \ddot{\phi}(t)$ is zero at least four times in each period of the curve. But there is no polynomial of degree > 1 with arbitrarily many zeroes on the real line.

Problem 5.

Let $f : S \to S$ be the smooth map from the unit cylinder to itself which is the restriction of the map

$$\begin{gathered}\mathbb{R}^3 \to \mathbb{R}^3 \\ (x, y, z) \mapsto \left(\frac{x^2 - y^2}{\sqrt{x^2 + y^2}}, \frac{2xy}{\sqrt{x^2 + y^2}}, z^2 \right)\end{gathered}$$

Equivalently, this is the map $(r\cos\theta, r\sin\theta, z) \mapsto (r\sin(2\theta), r\cos(2\theta), z^2)$.

(a) [5pts.] Choose an atlas for the unit cylinder and find the corresponding matrix representations of $D_{(1,0,3)}f$ and $D_{(0,1,0)}f$.

Solution: We take the usual parametrization of the unit cylinder $\sigma(u, v) = (\cos u, \sin u, v)$. For both points in this problem the surface patch $\sigma : U = (\frac{\pi}{2}, \frac{3\pi}{2})$ contains both $\mathbf{p}$ and $f(\mathbf{p})$. The map $U \to U$ induced by f is $(u, v) \to (2u, v^2)$. The Jacobian of this map is

$$J = \begin{pmatrix} 2 & 0 \\ 0 & 2v \end{pmatrix}$$

So with respect to the basis induced by the surface patch, the matrix of the map $D_{(1,0,3)}f : T_{(1,0,3)}S \to T_{(1,0,9)}S$ is

$$\begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$$

and the matrix of the map $D_{(0,1,0)}f : T_{(0,1,0)}S \to T_{(-1,0,0)}S$ is

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

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(b) [5pts.] Decide whether this map is a local diffeomorphism. After proving your answer, give a one-sentence geometric description of why or why not.

Solution: The derivative map $D_{(0,-1,0)}f$ is not an isomorphism, so f cannot be a local diffeomorphism. Geometrically, the map $z \to z^2$ is not injective near 0;

no point on the unit circle in the xy-plane has a neighborhood on which f is one-to-one. (The map folds the bottom half of the cylinder over onto the top half.